Week 5

Last time: Bounded linear operator

2.7.2 Lemma

Let T be a bounded linear operator. Then

① $\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\|=1}} \|T(x)\|$

② $\|T\|$ defines a norm on $B(X,Y)$

Pf ① $\{x \in \mathcal{D}(T) : \|x\|=1\} \subseteq \{x \in \mathcal{D}(T) : x \neq 0\}$

$$\Rightarrow \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\|=1}} \|T(x)\| = \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\|=1}} \frac{\|T(x)\|}{\|x\|} \le \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|} = \|T\|$$

For the other direction,

Let $x \in \mathcal{D}(T)$, $x \neq 0$,

$$\frac{\|T(x)\|}{\|x\|} = \left\| \frac{1}{\|x\|} T(x) \right\|$$

$$= \left\| T\left(\frac{x}{\|x\|}\right) \right\|$$

$$\le \sup_{\substack{y \in \mathcal{D}(T) \\ \|y\|=1}} \|T(y)\|$$

(Note $\left\| \frac{x}{\|x\|} \right\| = \frac{\|x\|}{\|x\|} = 1$)

$$\Rightarrow \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|} \le \sup_{\substack{y \in \mathcal{D}(T) \\ \|y\|=1}} \|T(y)\|$$

$\Rightarrow$ ①

2 Show N1 - N4

N1 $\|T\| \geq 0$ clearly from definition

N2 $\|T\| = 0 \Leftrightarrow T = 0$?

Check; $\|T\| = 0$

$$\Leftrightarrow \frac{\|T(x)\|}{\|x\|} = 0 \quad \forall x \neq 0$$

$$\Leftrightarrow \|T(x)\| = 0 \quad \forall x \neq 0$$

$$\Leftrightarrow T(x) = 0 \quad \forall x \neq 0$$

$$\Leftrightarrow T = 0$$

zero transformation

or zero operator

N3 $\|\alpha T\| = |\alpha| \|T\|$ Exercise

N4 Let $T_1, T_2 \in B(X, Y)$

For any $x \in X$, $\|x\| = 1$

$$\|(T_1 + T_2)(x)\| = \|T_1(x) + T_2(x)\|$$

$$\leq \|T_1(x)\| + \|T_2(x)\|$$

$$\leq \|T_1\| + \|T_2\|$$

Take sup over all $\|x\| = 1$

$$\Rightarrow \|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$$

Rmk We also just showed that $B(X, Y)$ is closed under addition and scalar multiplication

Examples of linear operators

(1) X is normed space

Identity operator

$$I_X : X \to X$$

$$I_X(x) = x$$

$$\|I_X\| = 1$$

(2) Zero operator has norm 0

(3) Differentiation operator

Let X be normed space of polynomials

$$\|p(x)\| = \max_{0 \le x \le 1} |p(x)| \quad \text{norm on } X$$

Define $T : X \to X$

$$(T(p))(t) = p'(t)$$

Consider $p_n(t) = t^n$

Then $(T(p_n))(t) = n t^{n-1}$

$$\|p_n(t)\| = \max_{0 \le t \le 1} |t^n| = 1$$

$$\|T(p_n)(t)\| = \max_{0 \le t \le 1} |n t^{n-1}| = n$$

$$\|T(p_n)(t)\| = \underset{\substack{\uparrow \\ \text{can be arbitrarily large}}}{n} \|p_n(t)\|$$

$\Rightarrow$ T is unbounded.

Continuity of bounded linear operator

Recall: Let X, Y be metric space and $\mathcal{D}(T) \subset X$

$T : \mathcal{D}(T) \to Y$ be a map

Defn T is continuous at $x_0 \in \mathcal{D}(T)$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that

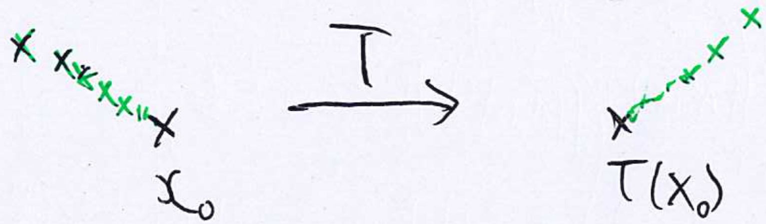
$$d(T(x), T(x_0)) < \varepsilon \text{ when } d(x, x_0) < \delta$$

T is said to be continuous if T is continuous at every $x \in \mathcal{D}(T)$

Important fact

T is continuous at $x_0 \in \mathcal{D}(T)$

$\Leftrightarrow$ For any sequence (x_n) with $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} T(x_n) = T(x_0)$



Thm 2.7-8 Let X be a normed space. $\dim X < \infty$, then every linear operator on X is bounded.

Pf Let $\dim X = n$ and $\{v_1, v_2, \cdots, v_n\}$ be a basis of X.

By lemma 2.4-1, $\exists\, c > 0$ s.t.

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \geq c \left(\sum_{i=1}^{n} |a_i| \right) \text{ for any } a_i$$

Let $x \in X$ and $x = \sum_{i=1}^{n} a_i v_i$, then

$$\begin{aligned}
\|T(x)\| &= \left\| \sum_{i=1}^{n} a_i T(v_i) \right\| \\
&\leq \sum_{i=1}^{n} \| a_i T(v_i) \| \\
&= \sum_{i=1}^{n} |a_i| \, \| T(v_i) \| \\
&\leq \sum_{i=1}^{n} |a_i| \left(\max_j \| T(v_j) \| \right) \\
&= \left(\max_j \| T(v_j) \| \right) \sum_{i=1}^{n} |a_i| \\
&\leq \left(\max_j \| T(v_j) \| \right) \frac{\|x\|}{c}
\end{aligned}$$

$\Rightarrow$ T is bounded

Thm 2.7-9 X, Y normed space

$T: \mathcal{D}(T) \subseteq X \to Y$ is linear.

Then TFAE

(i) T is continuous at a point $\in \mathcal{D}(T)$

(ii) T is continuous

(iii) T is bounded

Pf (i) $\Rightarrow$ (ii)

Assume T is continuous at $a \in \mathcal{D}(T)$

Let $b \in \mathcal{D}(T)$. and $x_n \in \mathcal{D}(T)$

such that $\lim_{n\to\infty} x_n = b$

Want to show $T(x_n) \to T(b)$

Consider the sequence $y_n = x_n - b + a$

Then
$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}(x_n - b + a) = \left(\lim_{n\to\infty} x_n\right) - b + a = b - b + a = a$$

T is continuous at a

$$\Rightarrow \lim_{n\to\infty} T(y_n) = T(a)$$

Since $y_n = x_n - b + a$, $T(y_n) = T(x_n) - T(b) + T(a)$

$$\Rightarrow T(x_n) = T(y_n) + T(b) - T(a)$$

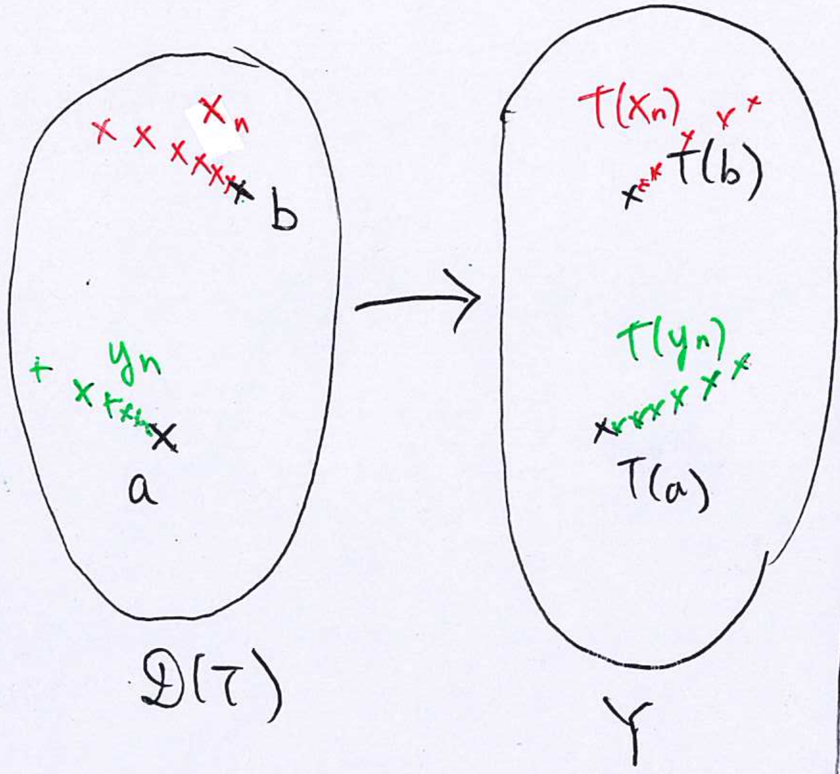
$$\lim_{n\to\infty} T(x_n) = \left(\lim_{n\to\infty} T(y_n)\right) + T(b) - T(a) = T(a) + T(b) - T(a) = T(b)$$

$\Rightarrow T$ is continuous at b

b is arbitrary in $\mathcal{D}(T)$

$\Rightarrow T$ is continuous

Picture:



(ii) $\Rightarrow$ (iii) Assume T is continuous

In particular, T is continuous at $\vec{0}$

Take $\varepsilon = 1$, $\exists\, \delta$ s.t. for any $x \in \mathcal{D}(T)$ with

$$\boxed{\underbrace{\|x - \vec{0}\|}_{= \|x\|} < \delta, \text{ then } \underbrace{\|T(x) - T(\vec{0})\|}_{= \|T(x)\|} < \varepsilon = 1 \quad (*)}$$

To show T is bounded, let $x \in \mathcal{D}(T)$, $\|x\| = 1$

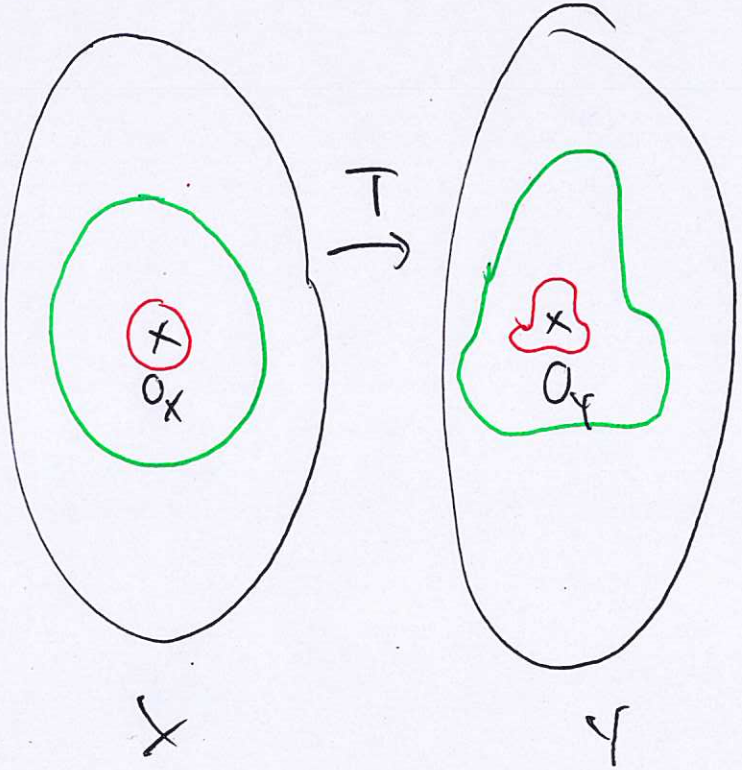
Consider $y = \frac{\delta}{2} x$, $\|y\| = \|\frac{\delta}{2} x\| = \frac{\delta}{2}\|x\| = \frac{\delta}{2}$

$(*) \Rightarrow \|T(y)\| < 1 \Rightarrow \|\frac{\delta}{2} T(x)\| < 1$

$\Rightarrow \|T(x)\| < \frac{2}{\delta}$

$\Rightarrow T$ is bounded

Picture



iii $\Rightarrow$ i Assume T is bounded

Want to show T is continuous at $\vec{O}_X \in \mathcal{D}(T)$

T is bounded

$\Rightarrow$ $\|T(x)\| \leq \|T\| \|x\|$ for all $x \in \mathcal{D}(T)$

Let $x_n \in \mathcal{D}(T)$ with $\lim_{n \to \infty} x_n = \vec{O}_X$

Then

$$0 \leq \|T(x_n) - \vec{O}_Y\| = \|T(x_n)\|$$
$$\leq \|T\| \|x_n\|$$

$$\lim_{n\to\infty} \|T\| \|x_n\| = \|T\| \lim_{n\to\infty} \|x_n\| = \|T\| \cdot 0 = 0$$

Sandwich thm $\Rightarrow$ $\lim_{n\to\infty} \|T(x_n) - \vec{O}_Y\| = 0$

$\Rightarrow$ $\lim_{n\to\infty} T(x_n) = \vec{O}_Y = T(\vec{O}_X)$

$\Rightarrow$ T is continuous at $\vec{O}_X$.

Defn Let X, Y be normed space,

Let $B \subset \mathcal{D}(T) \subset M \subset X$

$T : \mathcal{D}(T) \to Y$ be linear operator

The restriction of T to B is

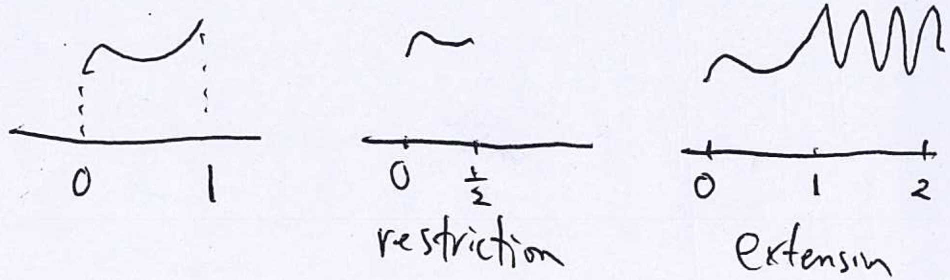
$$T|_B : B \to Y \quad \text{where}$$

$$T|_B(x) = T(x) \quad \forall x \in B$$

An extension of T to M is an linear operator $\tilde{T} : M \to Y$

such that $\tilde{T}(x) = T(x) \quad \forall x \in \mathcal{D}(T)$

eg (Not linear)



Thm 2.7-11 (Bounded linear extension)

Let Y be a Banach space, X be normed space

Let $T : \mathcal{D}(T) \to Y$ be a bounded linear operator, where $\mathcal{D}(T) \subseteq X$

Then T has an extension

$$\tilde{T} : \overline{\mathcal{D}(T)} \to Y$$

Here $\overline{\mathcal{D}(T)}$ is the closure of $\mathcal{D}(T)$ in X

Also, $\|\tilde{T}\| = \|T\|$ ($\tilde{T}$ is bounded)

Pf ① Define $\tilde{T}$:

Let $x \in \overline{\mathcal{D}(T)}$, then

$\exists x_n \in \mathcal{D}(T)$ such that $\lim_{n\to\infty} x_n = x$

Want to define $\tilde{T}(x) = \lim_{n\to\infty} T(x_n)$

Consider $\|T(x_n) - T(x_m)\| = \|T(x_n - x_m)\|$

$\underset{*}{\leq} \|T\| \|x_n - x_m\|$

We know (x_n) is convergent in X

$\Rightarrow (x_n)$ is Cauchy sequence

$\overset{*}{\Rightarrow} (T(x_n))$ is Cauchy

Y is Banach $\Rightarrow (T(x_n))$ is convergent in Y

Define $\tilde{T}(x) = \lim_{n\to\infty} T(x_n)$

② Check $\tilde{T}$ defined in step ① is well-defined

i.e $\tilde{T}(x)$ is independent of the chosen sequence x_n

Suppose $x_n' \in \mathcal{D}(T)$ such that $\lim x_n' = x$

Consider another sequence (v_n) defined by

$$(\underset{v_1}{x_1}, \underset{v_2}{x_1'}, \underset{v_3}{x_2}, \underset{v_4}{x_2'}, x_3, x_3', \cdots)$$

Then $\lim_{n\to\infty} v_n = x$

① $\Rightarrow \lim_{n\to\infty} T(v_n)$ exists

Note $T(x_n)$, $T(x_n')$ are subsequence of $T(v_n)$

Subsequence

$\Rightarrow \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} T(v_n) = \lim_{n\to\infty} T(x_n')$

$\Rightarrow \tilde{T}$ is well-defined

③ Show that $\tilde{T}(x) = T(x)$ for $x \in \mathcal{D}(T)$

Easy! If $x \in \mathcal{D}(T)$, consider constant sequence

$(x, x, x, x, \cdots)$

Then $\tilde{T}(x) = \lim_{n\to\infty} T(x) = T(x)$

④ Show that $\tilde{T}$ is linear.

Let $x, y \in \overline{\mathcal{D}(T)}$,

Let $x_n, y_n \in \mathcal{D}(T)$ with

$\lim_{n\to\infty} x_n = x \quad \lim_{n\to\infty} y_n = y$.

Then $\lim_{n\to\infty} x_n + y_n = x + y$

$$\begin{aligned}\tilde{T}(x+y) &= \lim_{n\to\infty} T(x_n + y_n) \\ &= \lim_{n\to\infty} \left(T(x_n) + T(y_n)\right) \\ &= \lim_{n\to\infty} T(x_n) + \lim_{n\to\infty} T(y_n) \\ &= \tilde{T}(x) + \tilde{T}(y)\end{aligned}$$

Similarly, one can show

$\tilde{T}(cx) = c\,\tilde{T}(x)$ for $c \in \mathbb{F}$

$\Rightarrow \tilde{T}$ is linear

⑤ Show $\|\tilde{T}\| = \|T\|$

Clearly $D(T) \subseteq D(\tilde{T})$

Also, $\tilde{T}(x) = T(x)$ on $\mathcal{D}(T)$

$\Rightarrow \|T\| \leq \|\tilde{T}\|$

Let $x \in \overline{\mathcal{D}(T)}$ and $x_n \in \mathcal{D}(T)$ with

$$\lim_{n\to\infty} x_n = x$$

Then $\tilde{T}(x) = \lim_{n\to\infty} T(x_n)$

Note $\|T(x_n)\| \leq \|T\| \|x_n\|$ --- ∗∗

Rmk Norm is a continuous function $\Rightarrow \lim_{n\to\infty} \|x_n\| = \|x\|$

∗∗ Take limit $n \to \infty$

$$\Rightarrow \|\tilde{T}(x)\| \leq \|T\| \|x\| \Rightarrow \frac{\|\tilde{T}(x)\|}{\|x\|} \leq \|T\|$$

for $x \neq 0$

$$\Rightarrow \|\tilde{T}\| \leq \|T\|$$

Hence $\|\tilde{T}\| = \|T\|$

Thm 2.10-2 (Completeness)

Let X be a normed space

Y be a Banach space

Then $B(X,Y)$ is Banach space

Pf is similar to 2.7-11